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Eigenvector Derivatives of Structures with Rigid Body Modes

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Introduction

BECAUSE of the increasing importance of sensitivity analysis in structural dynamics, development of effective and efficient methods for computing eigenvector derivatives has been an active area of research during the last three decades as shown in the excellent review paper by Haftka and Adelman.[1] The earliest method for computing eigenvector derivatives, apparently due to Fox and Kapoor,[2] requires all of the modes of a system and is computationally expensive. To improve computational efficiency, Nelson[3] developed an effective method to calculate eigenvector derivatives of the rth mode by just using the modal parameters of that mode. Application of Nelson's method, however, is limited to the case where only few eigenvector derivatives and few design parameters are of interest. To reduce computational effort involved in a wide range of applications in which a large number of eigenvector derivatives with respect to a large number of design parameters are required, an improvement to the truncated modal summation representation of eigenvector derivatives was proposed by Wang[4] in which a mode-acceleration type approach was used to obtain a static solution to approximate the contribution due to unavailable higher modes. In Ref. 5, an implicit method was further proposed by assuming that the eigenvector derivatives are spanned by the truncated mode shapes together with a residual static mode. Recently, Liu et al.[6] improved the accuracy of eigenvector derivatives of Refs. 4 and 5 by introducing higher order correction terms. Applications of the methods presented in Refs. 4–6, however, are practically limited since they are based on the prerequisite that the stiffness matrix of the structure to be analyzed is nonsingular and, hence, possesses no rigid body modes.

Since many practical structures possess rigid body modes due to insufficient structural constraints and some structural components are specifically analyzed under free-free conditions before they are coupled to form structural assemblies, this Note presents a new development that generalizes the methods of Refs. 4–6 so that structures with rigid body modes can be analyzed. By defining and solving a similar eigenvalue problem of a structure with rigid body modes, eigenvector derivatives can be accurately computed by using lower computed modes and a modified flexibility matrix. A numerical example is given to demonstrate the practicality of the proposed method.

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Existing Modal Superposition Methods

The matrix representation of vibration eigenvalue problem is

$$[K]\{\phi\}_r - \lambda_r[M]\{\phi\}_r = \{0\} \tag{1}$$

Differentiating with respect to design parameter p,

$$[[K]-\lambda_r[M]]\frac{\partial\{\phi\}_r}{\partial p}+\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}-\frac{\partial\lambda_r}{\partial p}[M]\right]\{\phi\}_r = 0 \tag{2}$$

Assume that $\{\phi\}_r$ is mass normalized such that

$$\{\phi\}_r^T[M]\{\phi\}_r = 1 \tag{3}$$

Without any loss of generality, the rth eigenvector derivatives can be expressed as[7]

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{s=1}^{N}\beta_{rs}\{\phi\}_s \tag{4}$$

Substituting Eq. (4) into Eq. (2) and premultiplying Eq. (2) by $\{\phi\}_s^T$, β_{rs} $(s \neq r)$ becomes

$$\beta_{rs} = \frac{1}{\lambda_r-\lambda_s}\{\phi\}_s^T\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}\right]\{\phi\}_r \qquad s\neq r \tag{5}$$

Upon differentiation of Eq. (3) and subsequent substitution of Eq. (4), β_{rr} can be obtained as

$$\beta_{rr} = -\frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r \qquad s = r \tag{6}$$

Therefore, the eigenvector derivatives of rth mode become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{N}\frac{1}{\lambda_r-\lambda_s}\{\phi\}_s^T\left[\frac{\partial[K]}{\partial p}-\lambda_r\frac{\partial[M]}{\partial p}\right]\{\phi\}_r\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{7}$$

Equation (7) represents the modal method proposed by Fox and Kapoor[2] that requires the availability of all of the modes of the system.

To reduce the computational cost, an improved modal method that aims to derive the required eigenvector derivatives approximately by using the calculated lower modes and the known flexibility matrix was developed by Wang.[4,5] Suppose that only few of the lower modes (m modes) of interest have been computed, then Eq. (7) can be modified to become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s + \sum_{s=m+1}^{N}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{8}$$

where $\{F\}_r \equiv [\partial[K]/\partial p - \lambda_r\partial[M]/\partial p]\{\phi\}_r$ and we have assumed that $r < m$. If the eigenvalues are numbered according to their magnitude in ascending order, then $\lambda_s - \lambda_r \approx \lambda_s$ for $s > r$ and Eq. (8) can be approximated as

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s + \sum_{s=m+1}^{N}\frac{\{\phi\}_s^T\{F\}_r}{-\lambda_s}\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{9}$$

The flexibility matrix $[K]^{-1}$, if it exists, can be written as

$$[K]^{-1} = [\phi][\lceil\lambda_\lrcorner]^{-1}[\phi]^T = \sum_{s=1}^{N}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s} \tag{10}$$

Upon substitution of Eq. (10), Eq. (9) becomes

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m} \frac{\{\phi\}_s^T\{F\}_r}{\lambda_r - \lambda_s}\{\phi\}_s - [K]^{-1}\{F\}_r + \sum_{s=1}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_s}\{\phi\}_s$$

$$-\frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r \tag{11}$$

which represents the improved modal method proposed by Wang.[4,5]

The improved modal method has recently been further developed by Liu et al.[6] by including higher correction terms. The second unknown term on the right-hand side (RHS) of Eq. (8) can be written as

$$\sum_{s=m+1}^{N}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s = -\sum_{k=0}^{\infty}\lambda_r^k\sum_{s=m+1}^{N}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s^{(k+1)}}\{F\}_r \tag{12}$$

in which the term $1/(\lambda_r - \lambda_s)$ has been expanded into convergent geometric series since $\lambda_r < \lambda_s$. Postmultiply both sides of Eq. (10) by $[M]$ and raise the power to $k+1$, then

$$[[K]^{-1}[M]]^{(k+1)} = [[\phi][\lceil\lambda_\lrcorner]^{-1}[\phi]^T[M]]^{(k+1)}$$

$$= [\phi][\lceil\lambda_\lrcorner]^{-(k+1)}[\phi]^T[M] \tag{13}$$

Upon separation of lower computed modes and higher uncomputed modes, Eq. (13) becomes

$$[[K]^{-1}[M]]^k[K]^{-1} = \sum_{s=1}^{m}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s^{(k+1)}} + \sum_{s=m+1}^{N}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s^{(k+1)}} \tag{14}$$

Upon substitution of Eqs. (14) and (12), Eq. (8) becomes

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\phi\}_s^T\{F\}_r}{\lambda_r-\lambda_s}\{\phi\}_s - \frac{1}{2}\{\phi\}_r^T\frac{\partial[M]}{\partial p}\{\phi\}_r\{\phi\}_r$$

$$-\sum_{k=0}^{\infty}\lambda_r^k\left[[[K]^{-1}[M]]^k[K]^{-1} - \sum_{s=1}^{m}\frac{\{\phi\}_s\{\phi\}_s^T}{\lambda_s^{(k+1)}}\right]\{F\}_r \tag{15}$$

Usually only the first few terms (up to the third) are required to obtain sufficiently accurate results. When only the first term ($k = 0$) is included, the method reduces to that of Wang.[4,5]

Analysis of Structures with Rigid Body Modes: New Development

Many practical structures possess rigid body modes due to insufficient structural constraints, and some structural components are specifically analyzed under free-free conditions before they are coupled to form structural assemblies. In the case when a structure possesses rigid body modes, its stiffness matrix $[K]$ becomes singular and, hence, its inverse $[K]^{-1}$ does not exist. Under such circumstances, the methods proposed by Wang[4,5] and Liu et al.,[6] which require the availability of $[K]^{-1}$, become inapplicable. To overcome such difficulty, a new method is developed and is described in this section. Before embarking on the detailed development of the method, it is necessary to define first the two systems that are similar in terms of their eigenproperties. The following eigenvalue problems are said to be similar:

$$[K]\{\phi\} - \lambda[M]\{\phi\} = \{0\} \tag{16}$$

$$[[K] + \mu[M]]\{\tilde{\phi}\} - \tilde{\lambda}[M]\{\tilde{\phi}\} = \{0\} \tag{17}$$

since they have the same set of eigenvectors and the eigenvalues are related by a single positive constant shift μ,

$$[\lceil\tilde{\lambda}_\lrcorner] = [\lceil\lambda + \mu_\lrcorner] \qquad [\tilde{\phi}] = [\phi] \tag{18}$$

To prove this, transform Eq. (16) as

$$[K]\{\phi\} - \lambda[M]\{\phi\} + \mu[M]\{\phi\} - \mu[M]\{\phi\} = \{0\} \tag{19}$$

which can be rewritten as

$$[[K] + \mu[M]]\{\phi\} - (\lambda + \mu)[M]\{\phi\} = \{0\} \tag{20}$$

Comparing Eq. (17) with Eq. (20), it can be seen that Eq. (18) can be established. Therefore, by solving the similar eigenvalue problem of Eq. (17), in which the stiffness matrix $[\tilde{K}] \equiv [K] + \mu[M]$ is nonsingular and, hence, Cholesky decomposition $[\tilde{K}] = [L][L]^T$ can be made, eigenvalues and eigenvectors of the original system (16) can be computed. From computational point of the view, solutions of eigenvalue problems of Eqs. (16) and (17) require the same amount of CPU time and memory storage. To compute eigenvector derivatives, however, solution of the similar eigenvalue problem becomes preferable and necessary.

Assume again that the first m modes are computed by solving partially the similar eigenvalue problem of Eq. (17), then by employing the relationships of Eq. (18), the eigenvector derivatives of Eq. (8) become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s + \sum_{s=m+1}^{N}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s$$

$$-\frac{1}{2}\{\tilde{\phi}\}_r^T\frac{\partial[M]}{\partial p}\{\tilde{\phi}\}_r\{\tilde{\phi}\}_r \tag{21}$$

The contribution of those higher uncomputed modes becomes

$$\sum_{s=m+1}^{N}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s = -\sum_{k=0}^{\infty}\tilde{\lambda}_r^k\sum_{s=m+1}^{N}\frac{\{\tilde{\phi}\}_s\{\tilde{\phi}\}_s^T}{\tilde{\lambda}_s^{(k+1)}}\{F\}_r \tag{22}$$

Following a similar procedure, as discussed, Eq. (22) can be further written as

$$\sum_{s=m+1}^{N}\frac{\{\tilde{\phi}\}_s\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s^T$$

$$= -\sum_{k=0}^{\infty}\tilde{\lambda}_r^k\left[[[\tilde{K}]^{-1}[M]]^k[\tilde{K}]^{-1} - \sum_{s=1}^{m}\frac{\{\tilde{\phi}\}_s\{\tilde{\phi}\}_s^T}{\tilde{\lambda}_s^{(k+1)}}\right]\{F\}_r \tag{23}$$

and the eigenvector derivatives, thus, become

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s - \frac{1}{2}\{\tilde{\phi}\}_r^T\frac{\partial[M]}{\partial p}\{\tilde{\phi}\}_r\{\tilde{\phi}\}_r$$

$$-\sum_{k=0}^{\infty}\tilde{\lambda}_r^k\left[[[\tilde{K}]^{-1}[M]]^k[\tilde{K}]^{-1} - \sum_{s=1}^{m}\frac{\{\tilde{\phi}\}_s\{\tilde{\phi}\}_s^T}{\tilde{\lambda}_s^{(k+1)}}\right]\{F\}_r \tag{24}$$

which is the general formula for the computation of eigenvector derivatives of structures with rigid body modes. From Eq. (24), it can be seen that only matrix–vector but no matrix–matrix multiplications are required in the computation of $\partial\{\phi\}_r/\partial p$. When only the first term is included, Eq. (24) becomes

$$\frac{\partial\{\phi\}_r}{\partial p} = \sum_{\substack{s=1\\s\neq r}}^{m}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_r-\tilde{\lambda}_s}\{\tilde{\phi}\}_s - \frac{1}{2}\{\tilde{\phi}\}_r^T\frac{\partial[M]}{\partial p}\{\tilde{\phi}\}_r\{\tilde{\phi}\}_r$$

$$-[\tilde{K}]^{-1}\{F\}_r + \sum_{s=1}^{m}\frac{\{\tilde{\phi}\}_s^T\{F\}_r}{\tilde{\lambda}_s}\{\tilde{\phi}\}_s \tag{25}$$

which is a modified version of Wang's method[4,5] but is applicable to the analysis of structures with rigid body modes. To summarize, the following becomes the complete procedure for the proposed method.

1) Define a modified stiffness matrix $[\tilde{K}] \equiv [K] + \mu[M]$.
2) Decompose $[\tilde{K}] = [L][L]^T$.
3) Perform the partial eigensolution: $[\tilde{K}]\{\tilde{\phi}\} - \tilde{\lambda}[M]\{\tilde{\phi}\} = \{0\} \Rightarrow [\lceil\tilde{\lambda}_{L\lrcorner}], [\tilde{\phi}_L]$.
4) Obtain $\partial\{\phi\}_r/\partial p$ based on Eq. (24).

Table 1 Eigenvector derivatives of the first nonrigid body mode

$\partial\{\phi\}/\partial p$	Exact	MTM	Error, %	Second order	Error, %	Third order	Error, %
$\partial v_{25}/\partial p$	−0.19963	−0.19156	4.042	−0.19751	1.062	−0.19877	0.431
$\partial v_{26}/\partial p$	−0.25849	−0.26086	0.917	−0.25993	0.569	−0.25902	0.205
$\partial v_{27}/\partial p$	−0.31716	−0.33067	4.260	−0.32004	0.908	−0.31851	0.426
$\partial v_{28}/\partial p$	−0.32978	−0.35281	6.983	−0.33215	0.691	−0.33110	0.400
$\partial v_{29}/\partial p$	−0.33872	−0.36874	8.863	−0.34391	1.532	−0.34058	0.549
$\partial v_{30}/\partial p$	−0.34387	−0.37796	9.914	−0.34897	1.431	−0.34587	0.582
$\partial v_{31}/\partial p$	−0.34516	−0.38018	8.900	−0.35301	2.274	−0.34761	0.710
$\partial v_{32}/\partial p$	−0.34256	−0.37530	9.557	−0.35012	2.467	−0.34603	1.013
$\partial v_{33}/\partial p$	−0.28435	−0.30514	7.311	−0.29219	2.757	−0.28729	1.034
$\partial v_{34}/\partial p$	−0.22664	−0.23613	4.187	−0.22791	0.560	−0.22705	0.181

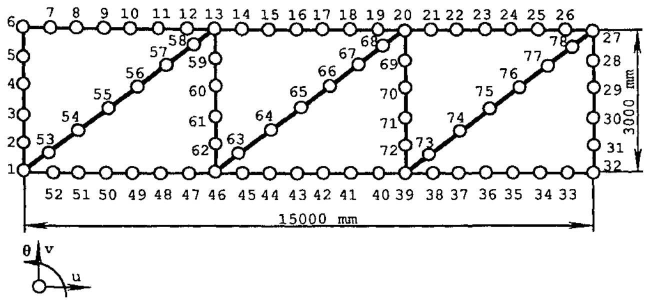

Fig. 1 Plane structure and its finite element mesh.

Numerical Example

To demonstrate its practical applicability, the proposed method has been applied to a structure shown in Fig. 1. The structure was modeled by 83 beam elements with three degree of freedom at each node. Young's modulus is assumed to be $E = 0.75 \times 10^{11}$ N/m^2 and density to be $\rho = 2800$ kg/m^3. The total number of degrees of freedom specified in the finite element model was 234. Without any loss of generality, Young's modulus of element 1 (between nodes 1 and 2) was chosen as design variable p and $\partial[K]/\partial p$ was computed. Since the structure has three rigid body modes, modified similar eigenvalue problem $[[K] + \mu[M]]\{\tilde{\phi}\} - \tilde{\lambda}[M]\{\tilde{\phi}\} = \{0\}$ was solved in which μ was arbitrarily chosen as $0.5\lambda_4$, where λ_4 is the first nonzero eigenvalue of the structure that is $\lambda_4 = 79{,}846(44.97$ Hz). Only the first six ($m = 6$) modes (the first three of them are rigid body modes) were solved, and it is assumed that the eigenvector derivatives of the first nonrigid body modes are of interest. Based on the proposed method, eigenvector derivatives of the first nonrigid body mode were computed, and some of the selected elements (eigenvector elements corresponding to v coordinates at nodes 25–34) are shown in Table 1 together with comparisons with those exact values and those from the mode truncation method (MTM) that is simply the modal method proposed by Fox and Kapoor[2] but with summation up to m available modes instead of all of the N modes of a system. As compared with their exact values, the second order ($k = 0$) and third order ($k = 0, 1$) approximations computed using the proposed method are quite accurate. This demonstrates the usefulness of the proposed method for eigensensitivity analysis of structures with rigid body modes.

Concluding Remarks

A new method for computing eigenvector derivatives of structures with rigid body modes is presented in this Note. The method can be considered as further development of the methods proposed by Wang[4,5] and Liu et al.[6] so that they become generally applicable to any practical structures. Such a generalization requires no extra computational cost yet enables the eigenvector derivatives of structures with rigid body modes to be computed with sufficient accuracy. A numerical example is given to demonstrate the practicality of the proposed method.

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Parametric Effects on Lift Force of an Airfoil in Unsteady Freestream

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Introduction

MANY flows in nature and technology involve cases where the freestream is unsteady. During the forward flight of a helicopter, the freestream velocity varies periodically accompanied by variations in angle of attack. Unsteady airfoils in pitching motion have been reviewed in detail by McCroskey.[1] The purpose of this Note is to report the effects of different operating parameters (reduced frequency, angle of attack, and amplitude of freestream variations) on the lift force of an airfoil in unsteady freestream.

It was shown, in a previous study,[2] that very high lift coefficients are observed depending on the frequency of the freestream variations. Other related studies[3–5] also suggested that other parameters such as angle of attack and amplitude of freestream variations

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